Subscription Dynamics and Competition in Communications Markets

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Outline

1. Introduction
2. Model
3. User Subscription Dynamics
   Equilibrium Analysis
   Convergence Analysis
4. Competition in Duopoly Markets
5. Illustrative Example
6. Conclusion
Overview of Communications Markets

Interaction among technology, users and service providers
How does the technology influence the users’ demand and the service providers’ revenues?

- We consider a duopoly communications market.
- Given prices, how does QoS affect the subscription decisions (or demand) of users?
- How are prices determined through competition between the service providers?
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Network model

- network service providers: $S_1$ and $S_2$
- continuum model: a large number of users
Model

Service providers

- $S_i$: price $p_i$ and fraction of subscribers $\lambda_i(p_i, p_{-i})$
Model

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- utility (revenue): $R_i(p_i, p_{-i}) = p_i \lambda_i(p_i, p_{-i})$
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Users

- user $k$: $u_k = \alpha_k q_i - p_i$ if it subscribes to $S_i$
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- user $k$: $u_k = \alpha_k q_i - p_i$ if it subscribes to $S_i$
- $\alpha_k$ follows a distribution with PDF $f(\alpha)$
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assumptions on $f(\alpha)$

• $f(\alpha) > 0$ if $\alpha \in [0, \beta]$ and $f(\alpha) = 0$ otherwise
• $f(\alpha)$ is continuous on $[0, \beta]$
Model

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QoS model

• $q_1$ is constant
• $q_2 = g(\lambda_2)$, where $g(\lambda_2) \in (0, q_1)$ is a differentiable and non-increasing function of $\lambda_2 \in [0, 1]$
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User Subscription

- Discrete-time model \( \{(\lambda_1^t, \lambda_2^t) \mid t = 0, 1, 2 \cdots \} \)

- Users' belief model and subscription decisions
  - naive (or static) expectation: every user expects that the QoS in the current period is equal to that in the previous period (i.e., \( \tilde{g}_k(\lambda_2^t) = g(\lambda_2^{t-1}) \))
  - a user subscribes to whichever NSP provides a higher (non-negative) utility
User Subscription

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- Dynamics of user subscriptions
User Subscription Dynamics

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- Dynamics of user subscriptions

  if \( \frac{p_1}{q_1} > \frac{p_2}{g(\lambda_2^{t-1})} \), then

\[
\lambda_1^t = h_{d,1}(\lambda_1^{t-1}, \lambda_2^{t-1}) = 1 - F \left( \frac{p_1 - p_2}{q_1 - g(\lambda_2^{t-1})} \right),
\]

\[
\lambda_2^t = h_{d,2}(\lambda_1^{t-1}, \lambda_2^{t-1}) = F \left( \frac{p_1 - p_2}{q_1 - g(\lambda_2^{t-1})} \right) - F \left( \frac{p_2}{g(\lambda_2^{t-1})} \right)
\]
User Subscription

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- Dynamics of user subscriptions

\[
\text{if } \frac{p_1}{q_1} \leq \frac{p_2}{g(\lambda^{t-1}_2)}, \text{ then }
\]

\[
\begin{align*}
\lambda^t_1 &= h_{d,1}(\lambda^{t-1}_1, \lambda^{t-1}_2) = 1 - F\left(\frac{p_1}{q_1}\right), \\
\lambda^t_2 &= h_{d,2}(\lambda^{t-1}_1, \lambda^{t-1}_2) = 0.
\end{align*}
\]
Equilibrium Analysis

- Stabilized fraction of subscribers will stabilize in the long run

**Definition**

$(\lambda_1^*, \lambda_2^*)$ is an *equilibrium* point of the user subscription dynamics in the duopoly market if it satisfies $h_{d,1}(\lambda_1^*, \lambda_2^*) = \lambda_1^*$ and $h_{d,2}(\lambda_1^*, \lambda_2^*) = \lambda_2^*$. 
Equilibrium Analysis

- Stabilized fraction of subscribers will stabilize in the long run

**Proposition (uniqueness and existence of \((\lambda_1^*, \lambda_2^*)\))**

For any non-negative price pair \((p_1, p_2)\), there exists a unique equilibrium point \((\lambda_1^*, \lambda_2^*)\) of the user subscription dynamics in the duopoly market. Moreover, \((\lambda_1^*, \lambda_2^*)\) satisfies

\[
\begin{align*}
\lambda_1^* &= 1 - F \left( \frac{p_1}{q_1} \right), \quad \lambda_2^* = 0, \quad \text{if} \quad \frac{p_1}{q_1} \leq \frac{p_2}{g(0)}, \\
\lambda_1^* &= 1 - F \left( \theta_1^* \right), \quad \lambda_2^* = F \left( \theta_1^* \right) - F \left( \theta_2^* \right), \quad \text{if} \quad \frac{p_1}{q_1} > \frac{p_2}{g(0)},
\end{align*}
\]

where \(\theta_1^* = (p_1 - p_2)/(q_1 - g(\lambda_2^*))\) and \(\theta_2^* = p_2/g(\lambda_2^*)\).
$q_1 = 2.5$, $g(\lambda_2) = 1.2e^{-0.5\lambda_2}$, and $\alpha$ is uniformly distributed on $[0, 1]$, i.e., $f_\alpha(\alpha) = 1$ for $\alpha \in [0, 1]$. 
Convergence of User Subscription Dynamics

- Convergence is not always guaranteed
Convergence of User Subscription Dynamics

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Example: when the QoS of NSP $S_2$ degrades fast w.r.t. the fraction of subscribers

1. suppose that only a small fraction of users subscribe to NSP $S_2$ at period $t$ and each subscriber obtains a high QoS
2. a large fraction of users subscribe at period $t+1$, which will result in a low QoS at period $t+1$
3. a small fraction of subscribers at period $t+2$
Convergence of User Subscription Dynamics

- Convergence is not always guaranteed

**Theorem**

For any non-negative price pair \((p_1, p_2)\), the user subscription dynamics converges to the unique equilibrium point starting from any initial point \((\lambda_1^0, \lambda_2^0) \in \Lambda\) if

\[
\max_{\lambda_2 \in [0,1]} \left\{ -\frac{g'(\lambda_2)}{g(\lambda_2)} \cdot \frac{q_1}{q_1 - g(\lambda_2)} \right\} < \frac{1}{K},
\]

where \(K = \max_{\alpha \in [0,\beta]} f(\alpha)\alpha\).
Illustration of Oscillation & Convergence

\[
\lambda_2: g(\lambda_2) = e^{-2\lambda_2}
\]

\[
\lambda_2: g(\lambda_2) = e^{-0.8\lambda_2}
\]

\[ t = 2 \]
Illustration of Oscillation & Convergence

$t = 3$

\[
\lambda_2: g(\lambda_2) = e^{-2\lambda_2}
\]

\[
\lambda_2: g(\lambda_2) = e^{-0.8\lambda_2}
\]
Illustration of Oscillation & Convergence

\[ \lambda_2(t) = e^{-\lambda_2 t} \]

\[ \lambda_2(t) = e^{-0.8\lambda_2 t} \]

\[ t = 4 \]
Illustration of Oscillation & Convergence

\[ \lambda_2(t) = e^{-0.8\lambda_2} \]

\[ \lambda_2(t) = e^{-2\lambda_2} \]

\[ t = 5 \]
Illustration of Oscillation & Convergence

\[ \lambda_2: g(\lambda_2) = e^{-0.8\lambda_2} \]

\[ \lambda_2: g(\lambda_2) = e^{-2\lambda_2} \]

\[ t = 6 \]
Illustration of Oscillation & Convergence

\[ \lambda_2: g(\lambda_2) = e^{-2\lambda_2} \]

\[ \lambda_2: g(\lambda_2) = e^{-0.8\lambda_2} \]

\[ t = 7 \]
Illustration of Oscillation & Convergence

\[ \lambda_2: g(\lambda_2) = e^{-2\lambda_2} \]

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\[ t = 15 \]
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We model competition between the NSPs using Cournot competition.

- each NSP chooses the fraction of subscribers independently
- prices are determined such that the equilibrium market shares equate the chosen quantities
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\[ G_C = \{ S_i, R_i(\lambda_1, \lambda_2), \lambda_i \in [0, 1) \mid i = 1, 2 \} \]
We model competition between the NSPs using Cournot competition.

- each NSP chooses the fraction of subscribers independently
- prices are determined such that the equilibrium market shares equate the chosen quantities

\[ \mathcal{G}_C = \{ S_i, R_i(\lambda_1, \lambda_2), \lambda_i \in [0, 1) \mid i = 1, 2 \} \]

\((\lambda_1^{**}, \lambda_2^{**})\) is a (pure) NE of \(\mathcal{G}_C\) (or a Cournot equilibrium) if it satisfies

\[ R_i(\lambda_i^{**}, \lambda_{-i}^{**}) \geq R_i(\lambda_i, \lambda_{-i}^{**}), \forall \lambda_i \in [0, 1), \forall i = 1, 2. \]
Existence of NE

Lemma

Suppose that $f(\cdot)$ is non-increasing on $[0, \beta]$. Let $\tilde{\lambda}_i(\lambda_{-i})$ be a market share that maximizes the revenue of NSP $S_i$ provided that NSP $S_{-i}$ chooses $\lambda_{-i} \in [0, 1)$, i.e., $\tilde{\lambda}_i(\lambda_{-i}) \in \arg\max_{\lambda_i \in [0,1)} R_i(\lambda_i, \lambda_{-i})$. Then $\tilde{\lambda}_i(\lambda_{-i}) \in (0, 1/2]$ for all $\lambda_{-i} \in [0, 1)$, for all $i = 1, 2$. Moreover, $\tilde{\lambda}_i(\lambda_{-i}) \neq 1/2$ if $\lambda_{-i} > 0$, for $i = 1, 2$. 
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• Implication
  • when the strategy space is specified as $[0, 1)$ and $f(\cdot)$ satisfies the non-increasing property, strategies $\lambda_i \in \{0\} \cup (1/2, 1)$ is strictly dominated for $i = 1, 2$
  • if a NE $(\lambda_1^*, \lambda_2^*)$ of $\tilde{G}_C$ exists, then it must satisfy $(\lambda_1^*, \lambda_2^*) \in (0, 1/2)^2$
Existence of NE

Theorem

Suppose that $f(\cdot)$ is non-increasing and continuously differentiable on $[0, \beta]$. If $f(\cdot)$ and $g(\cdot)$ satisfy some conditions (Eqn. 18 and Eqn. 19 in the paper), then the game $\tilde{G}_C$ has at least one NE.
Corollary

Suppose that the users’ valuation of QoS is uniformly distributed, i.e., \( f(\alpha) = \frac{1}{\beta} \) for \( \alpha \in [0, \beta] \). If \( g(\lambda_2) + \lambda_2 g'(\lambda_2) \geq 0 \) for all \( \lambda_2 \in [0, 1/2] \), then the game \( G_C \) has at least one NE.

- Interpretation
  - if the elasticity of the QoS provided by NSP \( S_2 \) with respect to the fraction of its subscribers is no larger than 1 (i.e., \( -[g'(\lambda_2)\lambda_2/g(\lambda_2)] \leq 1 \)), the Cournot competition game with the strategy space \([0, 1)\) has at least one NE.
  - the condition is analogous to the sufficient conditions for convergence in that it requires that the QoS provided by NSP \( S_2 \) cannot degrade too fast with respect to the fraction of subscribers.
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Figure: Dynamics of market shares under the best-response dynamics. Solid: $g(\lambda_2) = 1 - \frac{\lambda_2^2}{8}$; dashed: $g(\lambda_2) = 1 - \frac{\lambda_2^2}{2}$. 
Figure: Iteration of revenues under the best-response dynamics. Solid: $g(\lambda_2) = 1 - \frac{\lambda_2}{8}$; dashed: $g(\lambda_2) = 1 - \frac{\lambda_2}{2}$. 
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Conclusion

Study the impacts of technologies on the user subscription dynamics

- constructed the dynamics of user subscription based on myopic updates
- showed that the existence of a unique equilibrium point of the user subscription dynamics
- provided a sufficient condition for the convergence of the user subscription dynamics: the QoS provided by NSP $S_2$ should not degrade too fast as more users subscribe
Conclusion

Study the impacts of technologies on the user subscription dynamics

- constructed the dynamics of user subscription based on myopic updates
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Study the impacts of technologies on competition between the NSPs

- modeled the NSPs as strategic players in a non-cooperative Cournot game
- provided a sufficient condition that ensures the existence of at least one NE of the game
Selected References

Related Publications


Convergence of User Subscription Dynamics

Proof.

1. Show that

\[
\| h_d(\lambda_1, a, \lambda_2, a) - h_d(\lambda_1, b, \lambda_2, b) \|_\infty = K \left[ -\frac{g'(\lambda_2, c)}{g(\lambda_2, c)} \cdot \frac{q_1}{q_1 - g(\lambda_2, c)} \right] |\lambda_2, a - \lambda_2, b| \\
\leq \kappa_d \|\lambda_a - \lambda_b\|_\infty.
\]

where \( \kappa_d = K \cdot \max_{\lambda_2 \in [0, 1]} \left\{ \left[-\frac{g'(\lambda_2)}{g(\lambda_2)}\right] \cdot \left[\frac{q_1}{q_1 - g(\lambda_2)}\right] \right\} \)
Convergence of User Subscription Dynamics

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\[
\| h_d(\lambda_1, a, \lambda_2, a) - h_d(\lambda_1, b, \lambda_2, b) \|_\infty = K \left[ - \frac{g'(\lambda_2, c)}{g(\lambda_2, c)} \cdot \frac{q_1}{q_1 - g(\lambda_2, c)} \right] |\lambda_2, a - \lambda_2, b| \\
\leq \kappa_d \|\lambda_a - \lambda_b\|_\infty .
\]

where \( \kappa_d = K \cdot \max_{\lambda_2 \in [0,1]} \left\{ \left[ - \frac{g'(\lambda_2)}{g(\lambda_2)} \right] \cdot \frac{q_1}{q_1 - g(\lambda_2)} \right\} \)

2. If \( \max_{\lambda_2 \in [0,1]} \left\{ - \frac{g'(\lambda_2)}{g(\lambda_2)} \cdot \frac{q_1}{q_1 - g(\lambda_2)} \right\} < \frac{1}{K} \), then the mapping is contraction!