Liquidity in Credit Networks
A Little Trust Goes a Long Way

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(joint work with Ashish Goel, Ramesh Govindan, Ian Post)

NetEcon ’10
Vancouver, BC, Canada
Asterix willing to accept up to 100 IOUs from Obelix
Obelix willing to accept up to 50 IOUs from Asterix
Barter Economy in Armorica

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Barter Economy in Armorica

Obelix needs a favor worth 10 IOUs.

Gives Asterix 10 IOUs as payment.
Obelix needs a favor worth 10 IOUs.

Gives Asterix 10 IOUs as payment.
New trust values....
Barter Economy in Armorica
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PHEW! GLAD I DON'T OWE HIM ONE BACK…

ASTERIX, YOU OWE ME ONE!
What is a Credit Network?

• Decentralized payment infrastructure introduced by [DeFigueiredo, Barr, 2005] and [Ghosh et. al., 2007]
• Do not need banks, common currency
• Models trust in networked interactions
What is a Credit Network?

- **Graph** $G(V, E)$ represents a network (social network, p2p network, etc.)
  - **Nodes**: (non-rational) agents/players; print their own currency
  - **Edges**: credit limits $c_{uv} > 0$ extended by nodes to each other\(^1\)
  - Payments made by passing IOUs along a chain of trust
  - Credit gets replenished when payments are made in the other direction

\(^1\) assume all currency exchange ratios to be unity
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\[1\text{ assume all currency exchange ratios to be unity} \]
Applications

- Combating social spam (Facebook, LinkedIn)
- Distributing proxy addresses to circumvent censorship in repressive regimes
- Edges have integer capacity $c > 0$
- Transaction rate matrix $\Lambda = \{\lambda_{uv} : u, v \in V, \lambda_{uu} = 0\}$
- Repeated transactions; at each time step choose $(s, t)$ with prob. $\lambda_{st}$
- Try to route a unit payment from $s$ to $t$ via the shortest feasible path; **update edge capacities** along the path
- Transaction fails if no path exists
Markov Chain

- Repeated transactions induce a Markov chain $\mathcal{M}$ with $(c + 1)^m$ states
- State $S$ of $\mathcal{M}$ captures the states of all edges in $G$
- Transition probability $P(S, S') = \lambda_{st}$, where $s \rightarrow t$ in $S$ leads to $S'$
- $P(S, S)$: failure prob. at state $S$

Questions

- Steady-state distribution?
- Steady-state transaction success probability?
- Comparison with a centralized payment infrastructure
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- Steady-state transaction success probability?
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Main Results

- Success probability independent of path along which transactions are routed
- For symmetric transaction rates, the success probability for
  - Complete Graphs: Goes to one with increase in network size or credit capacity.
  - $G_c(n, p)$ networks ($p > \ln n/n$): Goes to one with increase in one of $n, p$ or $c$ keeping the other two constant.
  - PA networks: Goes to one with increase in avg. node degree or credit capacity (independent of network size).
- Success probability in Complete graphs and Erdős-Rényi graphs only constant-factor worse than equiv. centralized payment system.
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- Success probability in Complete graphs and Erdős-Rényi graphs only constant-factor worse than equiv. centralized payment system.
**Definition**

Let $S$ and $S'$ be two states of the network. We say that $S'$ is **cycle-reachable** from $S$ if the network can be transformed from state $S$ to state $S'$ by routing a sequence of payments along feasible cycles (i.e. from a node to itself along a feasible path).

Transactions along a feasible cycle are “free”. 
**Theorem**

Let \((s_1, t_1), (s_2, t_2), \ldots, (s_T, t_T)\) be the set of transactions of value \(v_1, v_2, \ldots, v_T\) respectively that succeed when the payment is routed along the shortest feasible path from \(s_i\) to \(t_i\). Then the same set of transactions succeed when the payment is routed along any feasible path from \(s_i\) to \(t_i\).
Proof Sketch.
Proof by induction on $T$.

$S_k :=$ state of the network when transactions $(s_1, t_1), \ldots, (s_k, t_k)$ are routed along the shortest feasible path

$S'_k :=$ state of the network when not all of the transactions $(s_1, t_1), \ldots, (s_k, t_k)$ are routed along the shortest feasible path

From $S'_k$ undo transactions $(s_k, t_k), (s_{k-1}, t_{k-1}), \ldots, (s_1, t_1)$ and redo $(s_1, t_1), \ldots, (s_k t_k)$ along their shortest feasible paths. This results in state $S$.

But undoing and redoing is equal to $k$ transactions along cycles. Therefore, $S_k$ and $S'_k$ are cycle reachable.

So if $(s_{k+1}, t_{k+1})$ is feasible in state $S_k$, it is also feasible in state $S'_k$.  

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Cycle-reachability induces a partition $\mathcal{C}$ on the set of states in $\mathcal{M}$.

**Fact**
For any equivalence class $C \in \mathcal{C}$, if a transaction $(s, t)$ is feasible in some state $S \in C$, it is feasible for all states $S' \in C$ (since $S$ is cycle-reachable from $S'$).

**Fact**
If a transaction $(s, t)$ is feasible in two states $S_i, S_j \in C$ and results in transitions to states $S'_i$ and $S'_j$ respectively, then $S'_i$ and $S'_j$ are cycle-reachable (in other words, belong to the same equivalence class).

**Corollary**
If a transaction $(s, t)$ in some state in the equivalence class $C_i$ results in a transition to a state in equivalence class $C_j$, then the reverse transaction $(t, s)$ from any state in $C_j$ will result in a transition to a state in $C_i$. 
Cycle-reachability induces a partition \( C \) on the set of states in \( M \).

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Theorem
Consider a Markov chain $M_{S_0}$ starting in state $S_0$ induced by a symmetric transaction rate matrix $\Lambda$. Let $C_{S_0} \subseteq C$ be the set of equivalence classes accessible from $S_0$ under the regime defined by $\Lambda$. Then $M_{S_0}$ has a uniform steady-state distribution over $C_{S_0}$.
Proof.

\[ T_{ij} := \{(s, t) \mid s \rightarrow t \text{ in state } S \in C_i \text{ leads to state } S' \in C_j\} \]

Define transition probability between \( C_i, C_j \in C_{S_0} \) as

\[ P(C_i, C_j) = \sum_{(s,t) \in T_{ij}} \lambda_{st} \]

Since \((s, t) \in T_{ij} \iff (t, s) \in T_{ji}\) and \(\Lambda\) is symmetric, therefore \(P\) is a symmetric stochastic matrix.

\[ \implies \text{ uniform distribution over } C_{S_0} \text{ is stationary w.r.t. } P. \]

Corollary

If \( \mathcal{M} \) is an ergodic Markov chain induced by a symmetric transaction rate matrix \( \Lambda \), it has a uniform steady state distribution over \( C \).
**Proof.**

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**Corollary**

If \(M\) is an ergodic Markov chain induced by a symmetric transaction rate matrix \(\Lambda\), it has a uniform steady state distribution over \(C\).
Proof.

$T_{ij} := \{(s, t) \mid s \rightarrow t \text{ in state } S \in C_i \text{ leads to state } S' \in C_j\}$

Define transition probability between $C_i, C_j \in \mathcal{C}_{S_0}$ as

$$P(C_i, C_j) = \sum_{(s,t) \in T_{ij}} \lambda_{st}$$

Since $(s, t) \in T_{ij} \Leftrightarrow (t, s) \in T_{ji}$ and $\Lambda$ is symmetric, therefore $P$ is a symmetric stochastic matrix.

$\implies$ uniform distribution over $\mathcal{C}_{S_0}$ is stationary w.r.t. $P$. \qed

Corollary

If $\mathcal{M}$ is an ergodic Markov chain induced by a symmetric transaction rate matrix $\Lambda$, it has a uniform steady state distribution over $\mathcal{C}$. 
NEED A FAVOR FROM CACOFONIX
...!#@%...
Analysis
Centralized Payment Infrastructure
Analysis
Centralized Payment Infrastructure
Convert Credit Network $\rightarrow$ Centralized Model

$$\forall u, c_{ru} = \sum_v c_{vu}$$

$\implies$ Total credit in the system is conserved during conversion
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$\implies$ Total credit in the system is conserved during conversion
$\mathcal{M}$ has $\binom{mc+m-1}{m-1}$ states
\[ \mathcal{M} \text{ has } \binom{mc + m - 1}{m - 1} \text{ states} \]
**Theorem**

If $M$ is ergodic and $\Lambda$ is symmetric, then $M$ has a uniform steady-state distribution.

**Corollary**

If $M$ is ergodic and $\Lambda$ is symmetric, then the steady-state success probability is $c/(c+1)$. 
Theorem

If $\mathcal{M}$ is ergodic and $\Lambda$ is symmetric, then $\mathcal{M}$ has a uniform steady-state distribution.

Corollary

If $\mathcal{M}$ is ergodic and $\Lambda$ is symmetric, then the steady-state success probability is $\frac{c}{c + 1}$. 
## Liquidity Comparison

<table>
<thead>
<tr>
<th></th>
<th>Credit Network</th>
<th>Centralized System</th>
</tr>
</thead>
<tbody>
<tr>
<td>Star-network</td>
<td>$\Theta(1/c)$</td>
<td>$\Theta(1/c)$</td>
</tr>
<tr>
<td>Complete Graph</td>
<td>$\Theta(1/nc)$</td>
<td>$\Theta(1/nc)$</td>
</tr>
<tr>
<td>$G_c(n, p)^2$</td>
<td>$\Theta(1npc)$</td>
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</tr>
</tbody>
</table>

**Table:** Steady-state Failure Probability in Credit Network v/s Centralized System

\[^2\text{bankruptcy probability}\]


**Simulations**

**Setup**

- Repeated transactions on $G_c(n, p)$ and PA graphs.
- Stopping criterion: success-rate in consecutive time windows $\leq \epsilon$
- Studied effect of varying network size, network density, and credit capacity
- For each run, recorded following metrics:
  - Number of (weakly) connected components
  - Avg. path length of successful transactions
  - Number of “sink” / “source” nodes
- Averaged metrics over 100 runs
Simulations

Effect of Variation in Credit Capacity

\( n = 100; \ p = 0.10; \ d = 5 \)
Simulations

Effect of Variation in Credit Capacity

\( n = 100; \ p = 0.10; \ d = 5 \)
Simulations

Effect of Variation in Network Size

c = 1; p = 0.10; d = 5

$G(n,p)$
Pref. Attachment
$G(n,p)$ with np=const.
Simulations

Effect of Variation in Network Size

$c = 1; p = 0.10; d = 5$
Open Problems

- Effect of node failures on liquidity and how it varies with network topology
- Effect of non-zero payment routing fees on liquidity
- Endow nodes with rationality: how do nodes initialize and update trust values?

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Questions?
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