Estimating the Margin of Victory for Instant-Runoff Voting*

David Cary

* also known as Ranked-Choice Voting, preferential voting, and the alternative vote
Overview

- Why estimate?
- What are we talking about?
- Estimates
- Worst-case accuracy
- Real elections
- Conclusions
Why Estimate?

Trustworthy Elections

Risk-limiting audits

Margin of Victory
Why Estimate?

IRV Trustworthy Elections

IRV Risk-Limiting Audits

IRV Margin of Victory

(not feasible)
Why Estimate?

IRV Trustworthy Elections?

- IRV Risk-Limiting Audits
- IRV Margin of Victory (not feasible)
Why Estimate?

IRV Trustworthy Elections

IRV Risk-Limiting Audits

IRV Margin of Victory
(not feasible sometimes)

IRV Margin of Victory
Lower Bound
Proposals for IRV Risk-Limiting Audits

Risk-limiting audits for nonplurality elections

Overview

- Why estimate? because we can; to do risk-limiting audits

- What are we talking about?
  - What is Instant-Runoff Voting?
  - What is a margin of victory?

- Estimates

- Worst-case accuracy

- Real elections

- Conclusions
Model of Instant-Runoff Voting

- Single winner
- Ballot ranks candidates in order of preference.
- Votes are counted and candidates are eliminated in a sequence of rounds.
- In each round, a ballot counts as one vote for the most preferred continuing candidate on the ballot, if one exists.
- In each round, one candidate with the fewest votes is eliminated for subsequent rounds.
- Ties for elimination are resolved by lottery.
- Rounds continue until just one candidate is in the round. That candidate is the winner.
Consistent IRV Features

- Number of candidates ranked on a ballot:
  - require ranking all candidates
  - limit maximum number of ranked candidates
  - can rank any number of candidates

- Multiple eliminations:
  - required*, not allowed, or discretionary*

- Early termination:
  - tabulation stops when a winner is identified*

* may require an extended tabulation for auditing purposes
The *margin of victory* is
the minimum total number* of ballots
that must in some combination
be added and removed
in order for the set of contest winner(s)
to change with some positive probability.

* the number of added ballots, plus
  the number of removed ballots
Overview

- Why estimate? *because we can; to do risk-limiting audits*
- What are we talking about?
- **Estimates**
- Worst-case accuracy
- Real elections
- Conclusions
## Estimates for the Margin of Victory

<table>
<thead>
<tr>
<th>Method</th>
<th>Time</th>
<th>Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>Last-Two-Candidates upper bound</td>
<td>$O(1)$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>Winner-Survival upper bound</td>
<td>$O(C)$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>Single-Elimination-Path lower bound</td>
<td>$O(C^2)$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>Best-Path lower bound</td>
<td>$O(C^2 \log C)$</td>
<td>$O(C)$</td>
</tr>
</tbody>
</table>

(*$C$ = number of candidates*)
# Example IRV Contest

<table>
<thead>
<tr>
<th>Candidate</th>
<th>Round 1</th>
<th>Round 2</th>
<th>Round 3</th>
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</tr>
</thead>
<tbody>
<tr>
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</tr>
<tr>
<td>Adrian Adams</td>
<td>20</td>
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Candidates are in reverse order of elimination, with the winner first.
## Last-Two-Candidates Upper Bound

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Margin of Survival for Winner in round C – 1, the round with just the last two candidates.
## Winner-Survival Upper Bound

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</tr>
<tr>
<td>Margin of Survival for Winner</td>
<td>87</td>
<td>71</td>
<td>30</td>
<td>40</td>
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</table>

Smallest Margin of Survival for the Winner in the first \( C - 1 \) rounds.
**Vote Totals Not In Sequence By Value**

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## Single-Elimination-Path Lower Bound

<table>
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<tr>
<th>Margin of Single Elimination (MoSE)</th>
<th>Round 1</th>
<th>Round 2</th>
<th>Round 3</th>
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<th>Round 5</th>
</tr>
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<tbody>
<tr>
<td>15</td>
<td>15</td>
<td>5</td>
<td>30</td>
<td>40</td>
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</table>

Smallest Margin of Single Elimination in the first C – 1 rounds.
Single Elimination Path

Round 1
- candidates \{a, b, c, d, w\}
  - 15 = MoSE(1)
Round 2
- candidates \{b, c, d, w\}
  - 5 = MoSE(2)
Round 3
- candidates \{c, d, w\}
  - 30 = MoSE(3)
Round 4
- candidates \{d, w\}
  - 40 = MoSE(4)
Round 5
- candidate \{w\}
Single-Elimination Path Bottleneck

Edge weight = a limited capacity (a bottleneck)
for tolerating additions and removals of ballots, while still staying on the path.

- **Round 1**: candidates \{a, b, c, d, w\}  
  - 15 = MoSE(1)
- **Round 2**: candidates \{b, c, d, w\}  
  - 5 = MoSE(2)
- **Round 3**: candidates \{c, d, w\}  
  - 30 = MoSE(3)
- **Round 4**: candidates \{d, w\}  
  - 40 = MoSE(4)
- **Round 5**: candidate \{w\}
Exceeding a Bottleneck

Easy guarantee of same winner:
Stay on the single-elimination path

Round 1: candidates \{a, b, c, d, w\}
15 = MoSE(1)

Round 2: candidates \{b, c, d, w\}
5 = MoSE(2)

Round 3: candidates \{c, d, w\}
30 = MoSE(3)

Round 4: candidates \{d, w\}
40 = MoSE(4)

Round 5: candidate \{w\}

Different Winner
Different Winner
Different Winner
Path Bottleneck is the smallest individual bottleneck on the path = Single-Elimination-Path lower bound
Multiple Elimination as a Detour

Round 1: candidates \{a, b, c, d, w\}
15 = MoSE(1)

Round 2: candidates \{b, c, d, w\}
5 = MoSE(2)

Round 3: candidates \{c, d, w\}
30 = MoSE(3)

Round 4: candidates \{d, w\}
40 = MoSE(4)

Round 5: candidate \{w\}
Multiple Elimination of $k$ Candidates

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A usable multiple elimination, if combined vote total is still the smallest
### Margin of Multiple Elimination

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MoME(2, 2) = 112 – (46 + 41)
= 112 – 87
= 25
Multiple Elimination as a Detour

Round 1: candidates \{a, b, c, d, w\}  
15 = MoSE(1)

Round 2: candidates \{b, c, d, w\}  
5 = MoSE(2)

Round 3: candidates \{c, d, w\}  
30 = MoSE(3)

Round 4: candidates \{d, w\}  
40 = MoSE(4)

Round 5: candidate \{w\}

MoME(2, 2) = 25
Usable Multiple Eliminations

Which path has the largest path bottleneck?
Best-Path Lower Bound

• The largest path bottleneck ...
  • of all paths from round 1 to round C ...
    • that consist of usable multiple eliminations.

• A best path:
  • Guarantees the same winner
  • Maximizes tolerance for additions and removals among usable multiple elimination paths
Best-Path Lower Bound Algorithms

• $O(C^2 \log C)$ time to sort the vote totals within each round.

• The best path can be found in $O(C^2)$ time.
  • Using a bottleneck algorithm, which is …
  • A longest path algorithm for a weighted directed acyclic graph, but calculating the length as the minimum of its component parts, instead of the sum.
Estimate Relations

Single-Elimination-Path lower bound

\[ \leq \text{Best-Path lower bound} \]

\[ \leq \text{margin of victory} \]

\[ \leq \text{Winner-Survival upper bound} \]

\[ \leq \text{Last-Two-Candidates upper bound} \]
Early-Termination Estimates

• For tabulations that stop before C-1 rounds
  • when a candidate has a majority of the continuing votes
  • more than two candidates are in the round

• Accuracy is degraded
  • must allow for possible extreme behavior in the missing rounds of the tabulation.
Overview

- Why estimate? **to do risk-limiting audits**
- What are we talking about?
- Estimates – **quick**: $O(C^2 \log C)$ time
- **Worst-case accuracy**
- Real elections
- Conclusions
Asymptotic Worst-Case Accuracy

- Ratio with margin of victory is unbounded.
  
  \[
  \frac{\text{Winner-Survival Upper Bound}}{\text{Margin of Victory}} = \frac{\text{Margin of Victory}}{\text{Best-Path Lower Bound}}
  \]

- No estimate can do better if based only on tabulation vote totals.
Asymptotic Worst-Case Example

Identical Tabulation Vote Totals

$2^{C-3}$

Winner-Survival
Upper Bound

Margin of Victory

Best Path
Lower Bound

$2^{C-3}$

contest 1

contest 2
Asymptotic Worst-Case Example

Ballots Show Different Margins of Victory

Margin of Victory

Best Path Lower Bound

Winner-Survival Upper Bound

contest 1

contest 2

$2^{C-3}$
Overview

• Why estimate? to do risk-limiting audits
• What are we talking about?
• Estimates – quick: $O(C^2 \log C)$ time
• Worst-case accuracy – unbounded ratios
• Real elections
• Conclusions
Estimates for Real Elections

- Australia elections, 2010
  - national House of Representatives
  - 150 contests
- All California IRV contests since 2004
  - local, non-partisan elections
  - 53 contests
  - 36 from San Francisco, 2004-2011
    - 12 using early termination estimates
  - 17 from Alameda county, 2010: Berkeley, Oakland, and San Leandro
Evaluating Estimates

• There are many ways to analyze the data.
• What are relevant metrics?
• A full evaluation requires a context of:
  • specific risk-limiting audit protocols
  • profiles of audit differences.
• Look at:
  • best available lower bound and upper bound,
  • as a percentage of first-round votes.
• What is the distribution of estimates?
## Selected Stats

<table>
<thead>
<tr>
<th>Assessment</th>
<th>Australia</th>
<th></th>
<th>California</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Contests</td>
<td>150</td>
<td>100%</td>
<td>53</td>
<td>100%</td>
</tr>
<tr>
<td>Contests with LB &gt; 10%</td>
<td>85</td>
<td>57%</td>
<td>35</td>
<td>66%</td>
</tr>
<tr>
<td>Contests with LB &lt; 5%</td>
<td>28</td>
<td>19%</td>
<td>14</td>
<td>26%</td>
</tr>
<tr>
<td>Contests with LB &lt; 1%</td>
<td>2</td>
<td>1%</td>
<td>7</td>
<td>13%</td>
</tr>
<tr>
<td>Contests with LB=MoV=UB</td>
<td>71</td>
<td>47%</td>
<td>16</td>
<td>30%</td>
</tr>
<tr>
<td>and LB &lt; 5%</td>
<td>21</td>
<td>14%</td>
<td>4</td>
<td>8%</td>
</tr>
<tr>
<td>and LB &lt; 1%</td>
<td>1</td>
<td>1%</td>
<td>0</td>
<td>0%</td>
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<tr>
<td>Contests with UB/LB &gt; 2</td>
<td>34</td>
<td>23%</td>
<td>10</td>
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Australia Elections

Margin of Victory, Range of Best Estimates
Australia 2010 House of Representatives

Percent of First-Round Votes

Count of Contests by Increasing Best-Path Lower Bound
California Elections

Margin of Victory, Range of Best Estimates
California Local IRV Elections, 2004-2010
Overview

• Why estimate? **to do risk-limiting audits**
• What are we talking about?
• Estimates – **quick**: $O(C^2 \log C)$ time
• Worst-case accuracy – **unbounded ratios**
• Real elections – **some estimates useful, some need improvement**

• **Conclusions**
Conclusions

- Risk-limiting audits can use lower bounds for the margin of victory.
- Estimates can be quickly calculated from tabulation vote totals.
- Worst-case ratios with the margin of victory are unbounded.
- The Best-Path lower bound can be used for some risk-limiting audits, but some contests will need better estimates.
Thanks

- Members and associates of Californians for Electoral Reform (CfER)
  - especially Jonathan Lundell
- San Francisco Voting System Task Force
  - especially Jim Soper
- anonymous reviewers
  - for many suggestions for improving the paper
  - especially for the idea of the Winner-Survival upper bound