訊息工程學系
Department of Information Engineering

Mathematical Modeling of Competition in Sponsored Search Market

Jerry Jian Liu and Dah Ming Chiu
Department of Information Engineering
The Chinese University of Hong Kong

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Outline

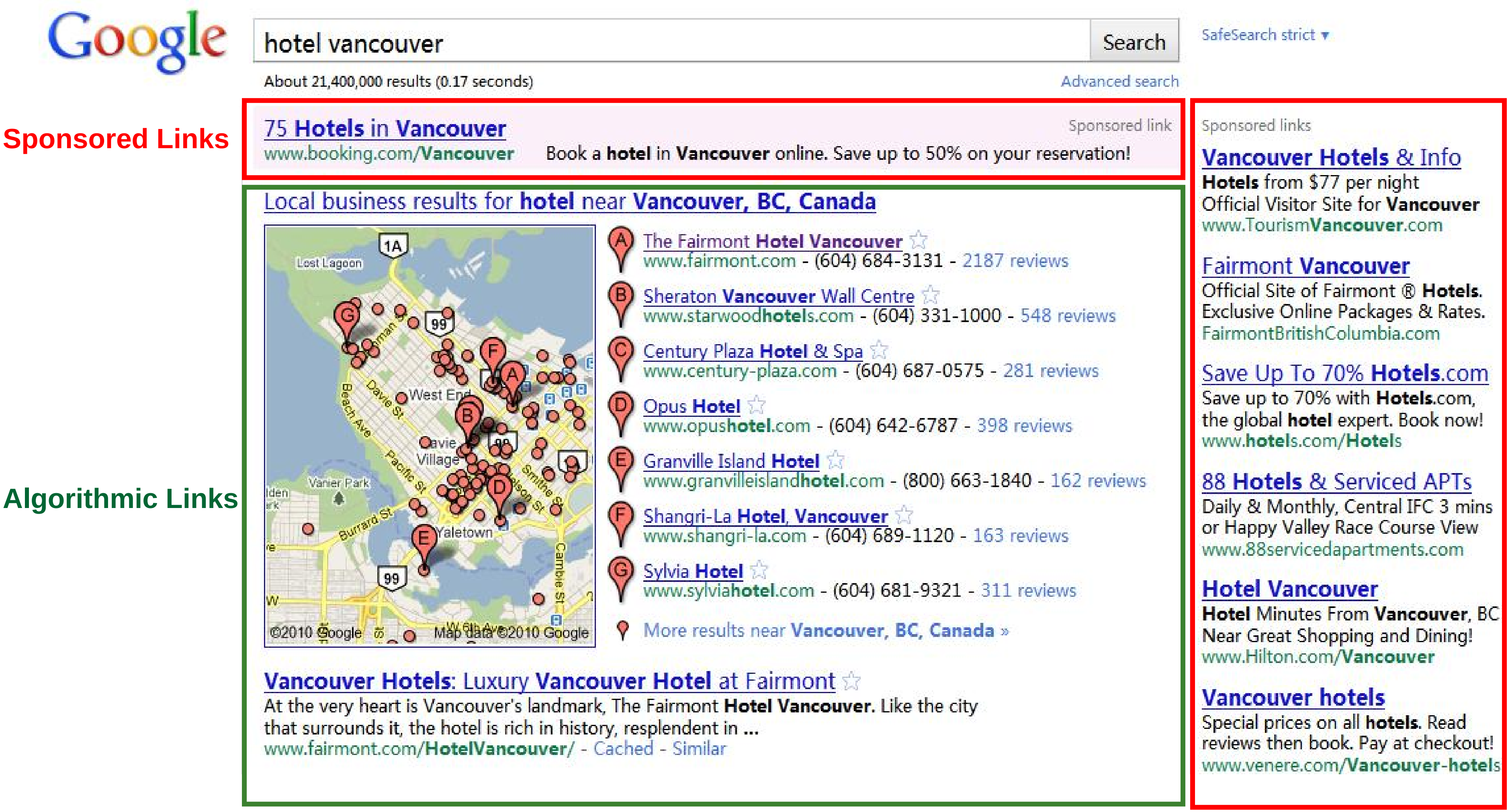
- Introduction
- The monopoly market model
- The duopoly market model
  - Competition for end users
  - Competition for advertisers
- Simulation results
- Summary

Background

- Internet advertising becomes a main source of revenue for primary search engines nowadays.

- Major search engines, like Google, Yahoo! and Microsoft all employ sponsored links to display advertisement when users submit their searching keywords.

Example of Sponsored Search



Motivation

- Most of previous works on sponsored search focused on mechanism design and analysis within the scope of one search engine.
- In practice, we notice that multiple search engines compete with each other for end users as well as advertisers in the market.
- How would the market evolve in the future? Will the leading company (like Google in US and Baidu in China) become the monopolist? Can small competitors still survive and co-exist?

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The Monopoly Market Model

- One search engine;
- A fixed set of end users;
- A fixed set of advertisers denoted by $\mathcal{I}\,(|\mathcal{I}| = m)$;
- Search engine can infer users' interest via the submitted keywords, and sell users' attentions to advertisers in the form of sponsored search.
- S: the supply of attentions for a particular keyword in a given time interval.
- Search engine needs to determine the price per attention to maximize its revenue:

$$R = p \cdot \min(S, D(p)) = \min(p \cdot S, pD(p))$$

Some explanations

$$R = p \cdot \min(S, D(p)) = \min(p \cdot S, pD(p))$$

- S: determined by end users.
- D: determined by advertisers.
- Regarded as an **auction** process:
  - Price starts from zero. All advertisers stay in the auction.
  - More demand than supply → price increases gradually.
  - More advertisers choose to quit, demand drops.
  - At the point when demand equals supply, items were cleared at that price.

Aggregate Demand

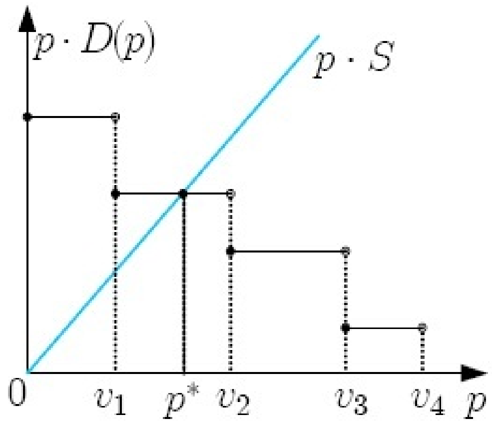
- In practice, each advertiser i would submit two parameters to the advertising system: **value** v_i for each attention and **budget** B_i in the given time interval.
- Reorder the index of advertisers such that $v_j \leq v_{j+1}$
- The aggregate demand is then:

$$D(p) = \sum_{i \in \mathcal{I}^+(p)} \frac{B_i}{p}$$

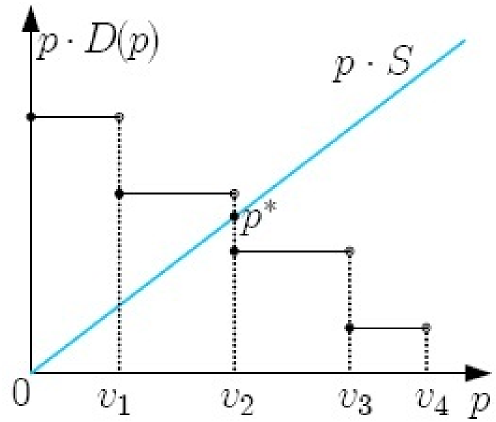
  where we define $\mathcal{I}^+(p) \triangleq \{i \in \mathcal{I} : v_i > p\}$.
- Thus, $p \cdot D(p) = \sum_{i \in \mathcal{I}^+(p)} B_i$ is also non-increasing over *p* since $\mathcal{I}^+(p)$ shrinks as price *p* increases. Furthermore, it's piece-wise constant.

Revenue as the Function of Price

- We can depict the revenue figures:



(a) Determined Advertisers



(b) Undetermined Advertiser

Search Engine Revenue Over Prices

  - For the undetermined advertiser scenario, we assume the search engine would allocate all the remaining supply to advertiser 2 as long as the current price doesn't exceed its value and the budget is not exhausted yet.

Optimal Price

- A polynomial step algorithm for calculating the optimal price:

Algorithm 1 Calculate Optimal Price $p^*(\mathcal{I})$

```
Begin
  v_0 = 0;
  for i = 1 : m
      sum = 0;
      for j = i : m
          sum += B_j;
      end for;
      p = sum/S;
      if (p ≤ v_i)
          return max(p, v_{i-1});
      end if;
  end for;
  return v_m;
End
```

Input:

(v_i, B_i) for each i

S of search engine

Output:

optimal p^*

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The Duopoly Market Model

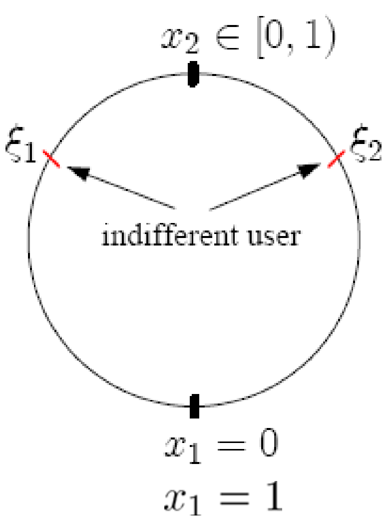
- Two horizontally and vertically differentiated search engines $\mathcal{J} = \{1, 2\}$ competing for users and advertisers.
- Horizontal difference means the different design of home pages and diversity of extra services like news, email.
  - Different users may have different tastes and preferences.
- Vertical difference means the quality of search results.
  - For users, the higher quality the better.
- We model the competition as a three-stage game:
  - Stage I, two engines provide various services to attract users;
  - Stage II, two engines determine their prices to advertisers;
  - Stage III, advertisers choose the engine which brings them higher utility.

Stage I: Classic Location Model

- Assuming users are spread uniformly on the circumference of a unit circle. Each user is characterized by an address *t* on the circle, denoting his specific taste.
- Each engine chooses a location in the characteristic space denoting the specific feature of service it provides.
- For user *t* searches at engine at location *x*, it will involve quadratic transportation cost $(t-x)^2$.
- Utility of user $t \in [0,1)$:

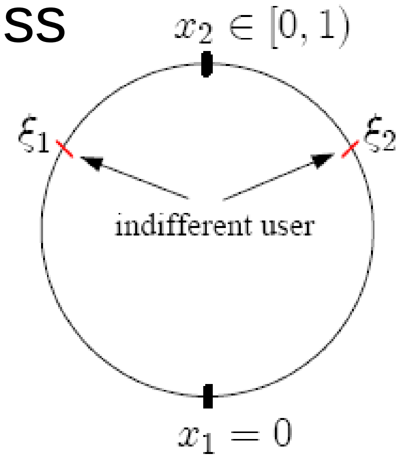
$$
\begin{aligned}
u_1(t) &= \zeta_1 q - C(t, x_1) = q - \min\{t^2, (1-t)^2\} \\
u_2(t) &= \zeta_2 q - C(t, x_2) = \zeta q - (t - x_2)^2
\end{aligned}
$$

  - $\zeta \in [0,1]$: vertical difference in quality;
  - $C(t,x)$: horizontal difference in design.



Division of User Market

- By letting $u_1(t) = u_2(t)$, we get the address of two indifferent users as ξ_1, ξ_2 .



- Then the market share of engine 2 is:

$$n_2(x_2) = \xi_2 - \xi_1$$

- By applying first-order condition $\frac{dn_2}{dx_2} = 0$, we can get the optimal address for engine 2:

$$x_2^* = \frac{1}{2}$$

i.e., the maximum differentiation.

Competition for Advertisers

- The utility of advertiser $i \in \mathcal{I}$ in either search engine is:

$$\pi_1^i = \max\{(v_i - p_1)\frac{B_i}{p_1}, 0\}$$

$$\pi_2^i = \max\{(v_i\rho_i - p_2)\frac{B_i}{p_2}, 0\}$$

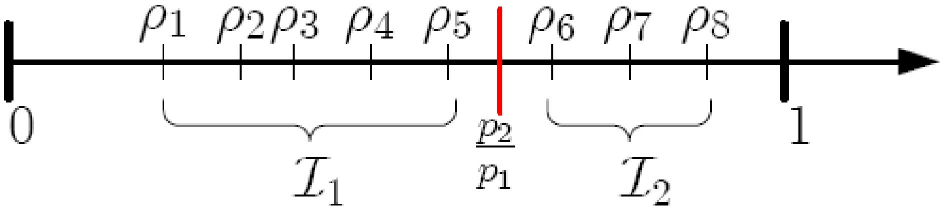
  - $\rho_i \in [0, 1]$ is a discount factor denoting advertiser i 's perceived “disability” of engine 2 to convert users’ attentions to clicks (or actual sales of products).
    - $\rho_i \approx 0$: more sensitive → Performance advertisers.
    - $\rho_i \approx 1$: less sensitive → Brand advertisers.

Division of Advertisers

- By letting $\pi_1^i \geq \pi_2^i$, we derive the condition under which advertiser *i* would choose engine 1:

$$\rho_i \leq \frac{p_2}{p_1}$$

- Reorder advertisers according to ρ_i , then the division of advertisers is as follows:



  - $\mathcal{I}_1(p_1, p_2) = \{i \in \mathcal{I} : \rho_i \leq \frac{p_2}{p_1}\}$: set of advertisers preferring engine 1;
  - $\mathcal{I}_2(p_1, p_2) = \{i \in \mathcal{I} : \rho_i > \frac{p_2}{p_1}\}$: set of advertisers preferring engine 2.

Nash Equilibrium Price Pair

- After initial price p_1 and p_2 are set in the market, advertisers are divided into $\mathcal{I}_1$ and $\mathcal{I}_2$. Each engine then compute its optimal price $p_1^*(\mathcal{I}_1)$ and $p_2^*(\mathcal{I}_2)$ independently as the monopoly case and price ratio p_2^*/p_1^* gets updated.
- If it happens the new ratio divides the advertisers into $\mathcal{I}_1$ and $\mathcal{I}_2$, then this is a Nash equilibrium price pair (p_1^{NE}, p_2^{NE})
- The formal **definition** is as follows:

  A price pair (p_1, p_2) is called *Nash equilibrium (NE) price pair* if $p_1 = p^*(\mathcal{I}_1(p_1, p_2))$ and $p_2 = p^*(\mathcal{I}_2(p_1, p_2))$ where $p^*(\mathcal{I})$ is calculated according to algorithm 1.

Existence of NE price pair

- **Theorem 1:** Assuming advertisers can *purchase service from both* search engines simultaneously, Nash equilibrium price pair would always exist for any set of advertisers and supplies of search engines.
  - The above assumption is necessary. Otherwise, the system would suffer from "oscillation" problem and no NE may exist.
  - A counter-example is when there is only one advertiser. No matter which engine it chooses, the price in the other engine is always zero. The advertiser would keep switching.
- **Theorem 2:** Denoted by (p_1^{NE}, p_2^{NE}) the NE price pair and p^* the optimal price when engine 1 monopolizes the market, it must hold that $p_2^{NE} \le p^* \le p_1^{NE}$.

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Four Major Criteria

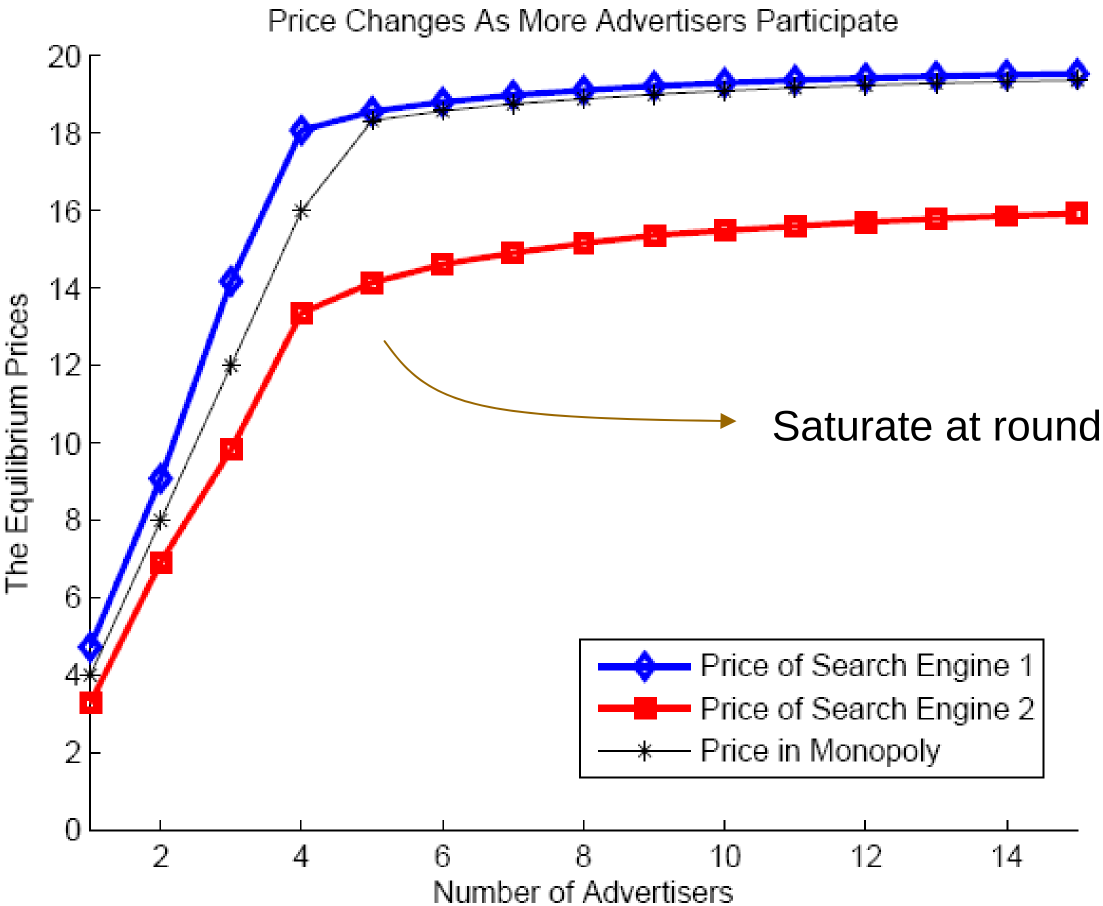
1. Prices: to compare the equilibrium prices with the monopoly price.
2. Revenues: to compare the total revenues under competition and monopoly. Merger or not?
3. Aggregate utility of advertisers: whether monopoly would harm the interests of advertisers.
4. Social welfare: the *realized value* of advertisers. Measure the interest of the community as a whole.

$$SW = \sum_{i \in \mathcal{I}_1} v_i q_{i1} + \sum_{i \in \mathcal{I}_2} \rho_i v_i q_{i2}$$

Baseline Setting

- We consider two search engines equally dividing the market. Suppose the total supply is normalized to one, the supplies of either engine is $S_1 = S_2 = 0.5$;
- Value v : uniformly distributed over (18, 20);
- Budget B : uniformly distributed with $E(B) = 4$;
- Discount factor ρ : uniformly distributed over (0.5,0.9) with expectation $E(\rho) = 0.7$.
  - To be exact, we define advertisers with $\rho \geq E(\rho)$ as brand advertisers.
  - The rest advertisers are all performance advertisers.

Prices in Baseline Setting

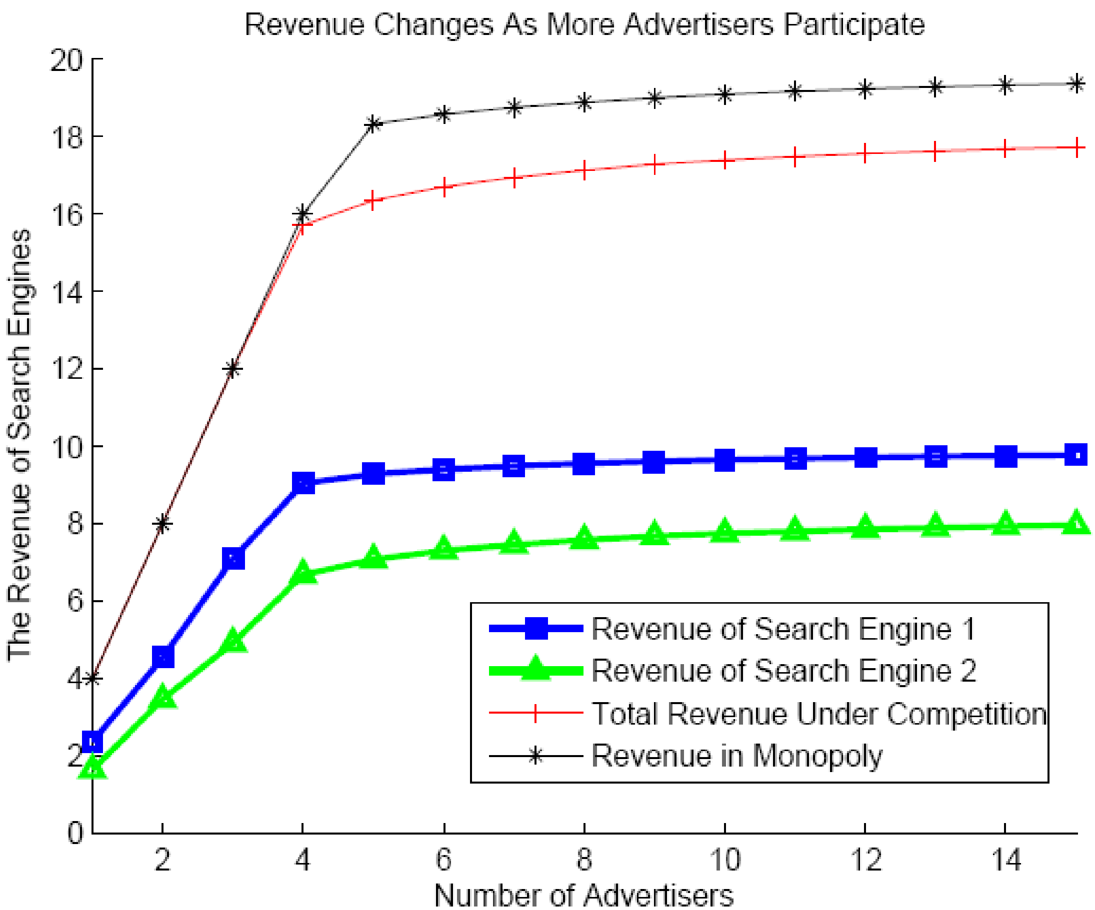


Approach maximal v

Approach maximal ρv

Saturate at round five since $E(B)/E(v) = 4/19 \approx 0.2$

Revenues in Baseline Setting



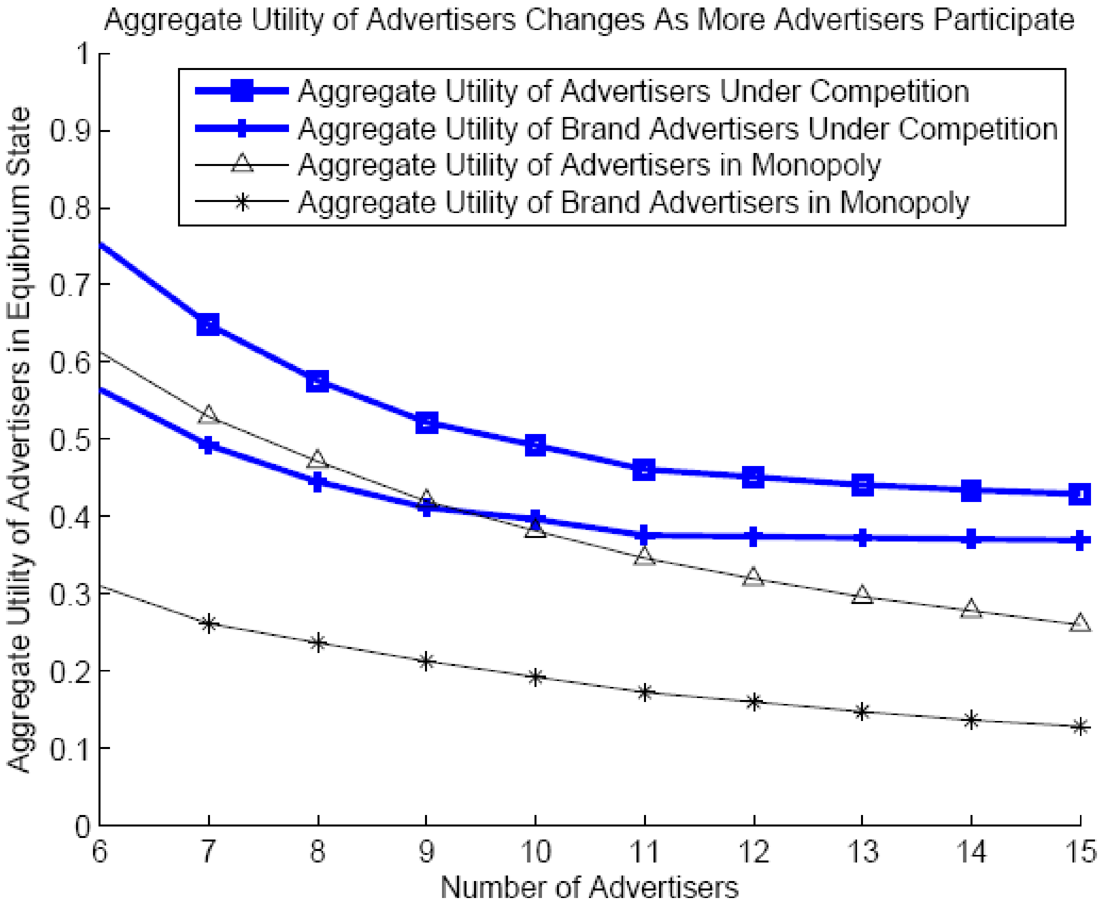
$$R = p^* \cdot (S_1 + S_2)$$

$$\approx R_1 + \frac{p_1}{p_2} R_2$$

$$R_1 = p_1 \cdot S_1$$

$$R_2 = p_2 \cdot S_2$$

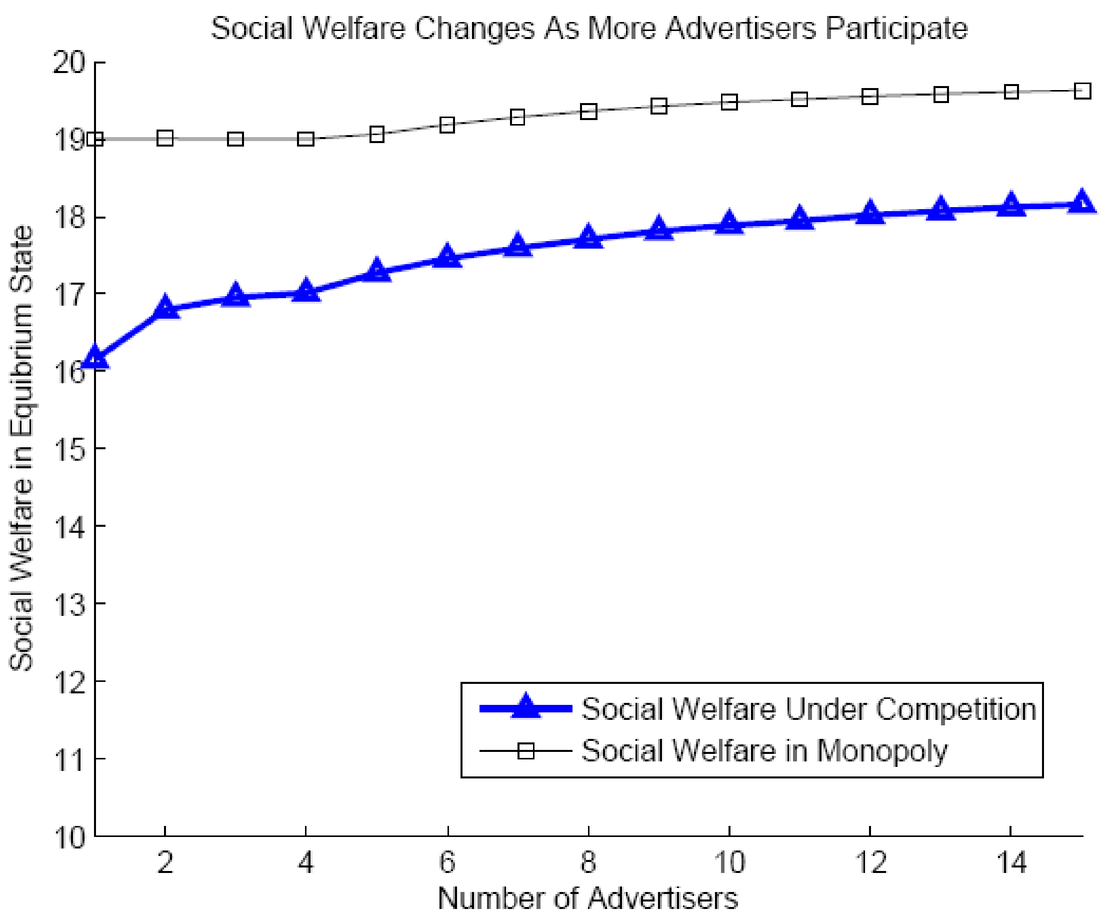
Aggregate Utility in Baseline Setting



Far more than half of the square line!
Brand advertisers have higher ρ and benefit from lower price of engine 2.

Exactly half of the triangular line!

Social Welfare in Baseline Setting



Realized values get discounted due to the effect of ρ :

$$SW = \sum_{i \in \mathcal{I}_1} v_i q_{i1} + \sum_{i \in \mathcal{I}_2} \rho_i v_i q_{i2}$$

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Summary of this work

- We propose an analytical framework to analyze the interaction of search engines, advertisers and end users in sponsored search market.
- A three-stage dynamic game is formulated to model the competition between search engines; furthermore, we prove the existence of Nash equilibrium of the game.
- We show some initial results of revenue and welfare of the advertising system by simulations.

Future Directions

- Throughout the work, we implicitly assume advertisers would reveal their true parameters. How would strategies of rational advertisers affect our conclusions?
- Associating our result of revenue from *one* keyword with practical scenario when revenue is aggregated from *numerous* keywords queried by different end users.
- Incorporating the generalized second-price (GSP) auction prevailing in major search engines.
- Investigating competition among multiple search engines analytically besides the duopoly scenario.

~The end~